SECOND ORDER DIFFERENTIAL EQUATIONS (theory)

Equation that has the form : $y^{(n)} = f(x)$

Order of these differential equation is reduce by the direct integration.

Equation that has the form : $F(x, y^{k}, y^{n}) = 0$

These equations deal with replacement: $y^{k} = p$, here is $y^{k+1} = p`$ etc. $(y` = p \longrightarrow y`` = p` \longrightarrow y``` = p``\ldots\ldots)$

Equation that has the form : $F(y, y`, y``, \ldots, y^{(n)})$

These equations deal with replacement: **y`=p**, but here we have to be careful, because: $y`` = p\dfrac{dp}{dy}$ , ( $y`` = p\,p`$)

Equation that has the form : $y`` + a(x)y` + b(x)y = f(x)$

Look at the appropriate homogeneous equation: **$y`` + a(x)y` + b(x)y = 0$**

If you know one particular solution $y_1(x)$ of this equation, then another solution we can find:

$y_2(x) = y_1(x)\int \dfrac{e^{-\int a(x)dx}}{y_1^{\,2}(x)}dx$, and the solution of homogeneous equation will be: $y(x) = c_1\, y_1(x) + c_2\, y_2(x)$

Then solve home inhomogenous equation by **undetermined coefficients** or by **variation of parameters** .

Euler equations

$$x^{n}y^{(n)} + a_1x^{n-1}y^{(n-1)} + \ldots + a_{n-1}xy` + a_ny = 0 \quad \text{or} \quad x^{n}y^{(n)} + a_1x^{n-1}y^{(n-1)} + \ldots + a_{n-1}xy` + a_ny = f(x)$$

Replacement: **$x = e^{t}$** ,and from here is: $y` = \dfrac{y`_t}{e^{t}}$; $y`` = \dfrac{y_t`` - y_t`}{e^{2t}}$; $y``` = \dfrac{y_t``` - 3y_t`` + 2y_t`}{e^{3t}}$ ……etc.

Because:

$$y` = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y`_t}{e^{t}} \longrightarrow y`` = \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{y`_t}{e^{t}}\right)}{\frac{dx}{dt}} = etc\ldots$$

First, we solve homogeneous Euler's equation, and then solve inhomogenous with **undetermined coefficients** or **variation of parameters** .

Linear homogeneous equation with constant coefficient (second order)

$$y'' + a_1 y` + a_2 = 0$$

First, write down the **characteristic equation:**

$$\lambda^2 + a_1\lambda + a_2 = 0$$

Depending on the characteristic equation solutions, we have three differentiate cases:

1) λ_1 and λ_2 are real and different, it is: $y(x)=c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x}$

2) λ_1 and λ_2 are real and equal solutions, it is: $y(x)= c_1e^{\lambda_1 x} + x\,c_2e^{\lambda_2 x}$

3) λ_1 and λ_2 are complex conjugate: λ_1=a+bi, λ_2=a-bi ,then: $y(x)=c_1e^{ax}\cos bx+c_2e^{ax}\sin bx$

Linear nonhomogeneous equation with constant coefficient (second order)

$$y'' + a_1 y` + a_2 = f(x)$$

First we solve homogeneous equation $y'' + a_1 y` + a_2 = 0$ and find $y=c_1(x)y_1+c_2(x)y_2$

1) **Method variation of parameters**

A system:

$$c`_1(x)y_1+c`_2(x)y_2 = 0$$

$$c`_1(x)y`_1+c`_2(x)y`_2 = f(x)$$

Solve the system " by c_1 and c_2 ", this solutions replace in $y = c_1(x)y_1+c_2(x)y_2$

Here we have to be careful, because: $c_1= c_1(x)$ and $c_2 = c_2(x)$

2) **Method undetermined coefficients**

I) If $f(x)=e^{ax}P_n(x)$ then:

- **i)** **a** is not the root of the characteristic equation, then $y=e^{ax}Q_n(x)$, where $Q_n(x)$ is polynomial of degree n with **undetermined coefficients**

- **ii)** **a** is the root of the characteristic equation , then $y= x^m e^{ax}Q_n(x)$, where m has same root order as **a**

II) If $f(x)= e^{ax}[P_n(x)\cos bx+Q_k(x)\sin bx]$ then:

- **i)** If **a** $\pm$ **bi** are not the roots of characteristic equation, then: $y = e^{ax}[S_N(x)\cos bx+T_N(x)\sin bx]$ where is N=max(n,k)

- **ii)** If **a** $\pm$ **bi** are the roots of characteristic equation , then: $y = x^m e^{ax}[S_N(x)\cos bx+T_N(x)\sin bx]$ where m has same root order as **a** $\pm$ **bi**